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ON THE WELFARE SIGNIFICANCE OF NATIONAL PRODUCT FOR ECONOMIC GROWTH AND SUSTAINABLE DEVELOPMENT*

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This paper formulates an investment value transversality condition in a continuous-time growth model, which characterizes competitive paths along which current net national product measures the welfare achieved along the path. This transversality condition requires that the present value of net investment goes to zero asymptotically. An example provided shows that, in general, competitive paths do not necessarily satisfy this condition. It is also shown that, in a standard growth model including an exhaustible resource as an essential factor of production, competitive paths always satisfy this condition. Implications regarding national income accounting procedures and sustainable development policies are discussed.
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1. Introduction

In recent years, there has been a renewal of interest in the concept of net national product (NNP). This interest has arisen from concerns regarding the exploitation of exhaustible and renewable natural resources, and the environmental effects of development. The focus of attention is on NNP as a measure of economic wellbeing, particularly the role of the net investment component of NNP in a measure of welfare in a dynamic economy and its connection with economic development policies.

It is well known that there is an index number problem if we seek to aggregate a basket of heterogeneous consumption goods using prices as weights in the aggregation. However, even if we abstract from this problem and suppose that there is an aggregate consumption index—a cardinal utility function which represents the welfare derived from consumption goods—a natural question arises in connection with the use of NNP as a measure of welfare. If we accept (following Samuelson, 1961) that the economic wellbeing of a nation, including its current and future generations, is properly represented by the present discounted value of future consumption, it is unclear why a current income concept such as NNP, which adds the value of net investment to current consumption, should be an appropriate measure of welfare. While the presence of the current consumption component is clearly appropriate, the justification for including current value of net investment is indirect at best, being related to future consumption by its influence on future productive potential. A central question then is whether a theoretical justification can be provided for this practice.

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Weitzman (1976) made the fundamental observation that, in theory, current net national product provides a precise measure of the present discounted value of current and future consumption (utilities). The observation is striking in two respects. The first is that a *current income* concept should contain all the information regarding the wellbeing of a society's entire future. This, of course, rests heavily on the fact that Weitzman makes his observation with respect to perfect-foresight competitive equilibrium paths where asset market equilibrating conditions link current prices to future prices, thereby encapsulating information about future resource scarcities in current-period prices. The second, and perhaps more intriguing, point is that, while it is to be expected that investment today translates in *some way* to the generation of future consumption, it is quite remarkable that the current value of net investment should turn out to be such an accurate proxy for the present discounted value of future consumption (utilities).

In this paper we take a critical look at Weitzman's analysis, shedding some light on the exact nature and scope of his fundamental proposition, as well as extending his analysis in some interesting directions. In order to understand properly Weitzman's basic proposition and its relationship to this paper, we have to conduct our discussion in more precise terms. Denote a *competitive path* by $(c(t), z(t), k(t), p(t))$, where $c(t)$, $z(t)$, $k(t)$ and $p(t)$ denote, respectively, consumption, net investments, capital stocks and prices of the capital goods in terms of the consumption good, at each time t. By a "competitive path" we mean a path along which the value of output is maximized and asset markets are in equilibrium, at each time t. (For precise definitions see Section 2.3 below.) Weitzman's observation is that, along a competitive path under a constant interest regime r, for every t,

$$c(t) + p(t)\dot{k}(t) = r\int_t^{\infty} \mathrm{e}^{-r(s-t)} c(s)\,\mathrm{d}s. \tag{1}$$

Let us refer to (1) as *Weitzman's Rule*. The left-hand side of (1) is the net national product (NNP), and thus the rule states that the NNP at time t is equal to the annuity equivalent of the present discounted value of consumption along the path.

If the competitive path $(c(t), z(t), k(t), p(t))$ were in fact an optimal path, that is, one that maximizes

$$\int_0^{\infty} \mathrm{e}^{-rt} c(t)\,\mathrm{d}t \tag{2}$$

over all paths from the same initial conditions, then Weitzman's Rule is really an interpretation of Bellman's equation of optimality in dynamic programming; viz., for all t,

$$-V'(k(t))\dot{k}(t) = c(t) - rV(k(t)), \tag{3}$$

where V is the "value function" associated with the problem of maximizing (2), assumed differentiable for convenience. The support of the value function, $V'(k(t))$, is the price vector $p(t)$ of the capital goods (for convex structures), and (3) readily yields (1).

However, in general, a competitive path is not optimal. It is optimal if and only if it satisfies the *capital value transversality condition*: $\lim_{t\to\infty} \mathrm{e}^{-rt} p(t)k(t) = 0$. The relation (1) would be of considerable interest if it turned out to be valid for competitive paths in general and not just for paths that are optimal according to a planner's problem. Indeed, the discussion in Weitzman (1976) suggests that he has such a proposition in mind, viz. that (1) holds for every competitive path.

Our analysis in this paper sheds light on the interesting question as to whether a rule like (1) is valid for competitive paths in general. We first make the following observation (see Theorem 1 and its Corollary in Section 3.3). A competitive path satisfies Weitzman's Rule if and only if it satisfies the *investment value transversality condition*,

$$\lim_{t\to\infty} \mathrm{e}^{-rt} p(t)\dot{k}(t) = 0. \tag{4}$$

Next, we use the characterization to show that the relation (1) is not valid for competitive paths in general. We do this by constructing a concrete example in the one-sector neoclassical growth model and exhibiting a competitive path that does not satisfy condition (4) (see Example 3 in Section 3.4). Finally, we observe (see Theorem 2 in Section 4.1) that, interestingly enough, in the standard model of optimal intertemporal allocation in which an exhaustible resource is a factor of production (see e.g. Dasgupta and Heal, 1974, 1979; Solow, 1974), the investment value transversality condition is *always* satisfied for competitive paths (even when the capital value transversality condition is not satisfied and consequently the path is not optimal). Thus, in this framework, Weitzman's Rule holds for every competitive path.

We end this introduction by drawing attention to two implications of Weitzman's Rule. There is an important implication that can be drawn for national income accounting procedures. Currently, national income accounts add consumption to the value of net investment in producible capital goods to arrive at the NNP. Since the stock of an exhaustible resource can be considered as a special type of capital good which cannot be augmented, national income accounts should reduce this calculated NNP by the value of exhaustible resources used up during the year (a disinvestment in these capital goods) in order that it be an appropriate indicator of the economic welfare of the nation. In other words, in order to reflect accurately the right-hand side of equation (1), one should calculate the left-hand side of (1), including investment in *all* capital goods, broadly defined (including exhaustible resource stocks).

The validity of Weitzman's Rule for all competitive paths also has an important implication for the discussion on investment policies ensuring *sustainable development*. Following Weitzman (1995), if we define a path of sustainable development to be one for which, at each date, the constant consumption equivalent of the present discounted value of future consumption is at least as large as the current consumption, then a competitive path generates a path of sustainable development if and only if the value of investment is never negative. This calls for an investment policy for sustainable development in which producible capital goods are augmented at a rate sufficient to offset the depletion of exhaustible natural resources, and produce a non-negative *aggregate* value of net investment.

2. Preliminaries

2.1 The framework

Consider a framework in which population and technology are unchanging, individuals are identical in all respects (so one can think in terms of a single representative person at each date and ignore distribution considerations) and, most importantly, consumption level in any period can be represented by a single number (denoted by c). "It might be calculated as an index number with given price weights, or as a multiple of some fixed basket of goods, or more generally as any cardinal utility function" (Weitzman, 1976, pp. 156–157). We shall adopt here the last interpretation in the preceding quote. Denote by C the non-negative vector of consumption flows $(C_1, \ldots, C_m)$ in $\mathbb{R}^m_+$, where C_j is the consumption of the jth consumption good, $j = 1, \ldots, m$. Denote by w: $\mathbb{R}^m_+ \to \mathbb{R}_+$ a cardinal welfare index, whose value at a consumption vector C will be called the (aggregate) consumption level and denoted by c; that is, $c \equiv w(C)$. From now on, when we refer to consumption we shall mean precisely the level of this cardinal utility index, i.e. c.

Denote by $k_i \geqslant 0$ the stock of the ith capital good, where $i = 1, \ldots, n$, and by z_i the investment flow, net of depreciation, of the ith capital good. Denote the vectors $(k_1, \ldots, k_n)$ and $(z_1, \ldots, z_n)$ by k and z, respectively. The technology set, denoted by Ω, is a set of triplets (c, z, k) in $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n_+$. A typical point (c, z, k) of Ω is understood to mean that from capital input stock k it is technologically feasible to obtain the flow of consumption c and the flows of net investments z. We shall first make the following assumption on Ω.[1]

(A1) (a) Ω is a closed and convex subset of $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n_+$; for each $k \geqslant 0$, there is a c in $\mathbb{R}$ and a z in $\mathbb{R}^n$ such that $(c, z, k) \in \Omega$.

(b) Given any number $\xi > 0$ there is a number $\eta > 0$ such that $(c, z, k) \in \Omega$ and $|k| \leqslant \xi$ implies $c \leqslant \eta$ and $|z| \leqslant \eta$.

(c) (i) if $(c, z, k) \in \Omega$ and c' satisfies $0 \leqslant c' \leqslant c$, then $(c', z, k) \in \Omega$.

(ii) if (c, z, k) and (c', z', k) are points in Ω satisfying $z' \leqslant z$, then $(c, z', k) \in \Omega$.

We need to remark only on (A1) (c) (ii); the rest of the assumption is standard and needs no explanation. Part (ii) of (A1) (c) is a form of free disposal assumption, just like part (i). It says that, given $(c, z, k) \in \Omega$, *if* it is possible to reduce the rate of investment z_i of some good i, then it is possible simultaneously to maintain consumption c.

We may alternatively represent the technological possibilities by associating with each k the combinations of consumption and net investment that it is possible to realize with input stocks k. Denote this by a set $S(k)$; that is,

$$S(k) \equiv \{(c, z): (c, z, k) \in \Omega\}. \tag{5}$$

Let Λ be the projection of the second and third components of Ω; that is,

$$\Lambda \equiv \{(z, k): (c, z, k) \in \Omega \text{ for some c}\}. \tag{6}$$

1) We are using conventional notation: for x, y in $\mathbb{R}^n$, $x \geqslant y$ means $x_i \geqslant y_i$ for $i = 1, \ldots, n$; $x > y$ means $x \geqslant y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for all $i = 1, \ldots, n$. For x in $\mathbb{R}^n$, the sum norm of x, denoted by $|x|$, is defined by $|x| = \sum_{i=1}^n |x_i|$. If $x(t)$ is a function of time, then $\dot{x}$ means time derivative of x. If $f: \mathbb{R}^n \to \mathbb{R}^m$, then $f'(x)$ is the $m \times n$ matrix whose ijth element is $(\partial f_i/\partial x_j)(x)$. If $f: \mathbb{R}^n \to \mathbb{R}$, then f_i is the ith partial derivative of f and f_{ij} is the jth partial derivative of f_i, $i = 1, \ldots, n$, $j = 1, \ldots, n$. The notation "a.e." stands for "almost everywhere"; more precisely, if A is a subset of $\mathbb{R}$, then by the expression "for $t \in A$, a.e." we mean "for $t \in B$, where B is a subset of A such that the complement of B in A is a set of Lebesgue measure zero"; if the set A is an interval $[a, \infty)$, we often use the expression "for $t \geqslant a$, a.e." in place of "for $t \in [a, \infty)$, a.e.".

Then Λ is the set of (z, k) pairs such that it is feasible to attain the rate of net investment z, along with *some* feasible consumption good output, given the capital input k.

Remark 1. *Under (A1), (i) for each* $k \geqslant 0$, $S(k)$ *is a non-empty, compact and convex subset of* $\mathbb{R}_+ \times \mathbb{R}^n$; *and (ii)* Λ *is a non-empty, closed and convex subset of* $\mathbb{R}^n \times \mathbb{R}^n_+$.

For each $(z, k) \in \Lambda$, the set $\{c: (c, z, k) \in \Omega\}$ is non-empty and compact, by (A1). We may, therefore, define a function $u: \Lambda \to \mathbb{R}_+$ as follows: for each $(z, k) \in \Lambda$,

$$u(z, k) \equiv \max\{c: (c, z, k) \in \Omega\}. \tag{7}$$

Thus, $u(z, k)$ is the maximum consumption good output that can be obtained from the capital input k, given output z of the investment goods.

Remark 2. *Under (A1), (i)* u *is an upper semicontinuous and concave function; (ii)* $u(z, k) \geqslant 0$ *for* $(z, k) \in \Lambda$; *and (iii)* u *is non-increasing in* z; *i.e.* $u(z, k) \geqslant u(z', k)$ *if* (z, k) *and* $(z', k) \in \Lambda$ *and* $z \leqslant z'$. *Part (iii) of Remark 2 follows from part (ii) of (A1)(c).*

We shall assume that u satisfies stronger properties.

(A2) (i) u is continuous on Λ and twice continuously differentiable in the interior of Λ.

(ii) For each $k \gg 0$, $u(z, k)$ is a strictly concave function of z; that is, if (z', k) and (z, k) are in Λ satisfying $z' \neq z$ and λ is a number satisfying $0 < \lambda < 1$, then $u[\lambda z + (1 - \lambda)z', k] > \lambda u(z, k) + (1 - \lambda)u(z', k)$; in the interior of Λ, the matrix of second partials of u with respect to z, $(\partial^2 u/\partial z^2)(z, k)$ is negative definite.

A *path* from initial stock K in $\mathbb{R}^n_+$ is a triplet of functions $(c(\cdot), z(\cdot), k(\cdot))$, where $c(\cdot)$: $[0, \infty) \to \mathbb{R}_+$, $z(\cdot)$: $[0, \infty) \to \mathbb{R}^n$ and $k(\cdot)$: $[0, \infty) \to \mathbb{R}^n_+$, such that $k(\cdot)$ is absolutely continuous and

$$(c(t), z(t)) \in S(k(t)) \text{ for } t \geqslant 0, \text{ a.e.}; \ \dot{k}(t) = z(t) \text{ for } t \geqslant 0, \text{ a.e.}; \ k(0) = K. \tag{8}$$

Denote by $\mathfrak{F}(K)$ the set of paths from initial stock K. Assume also the following.

(A3) (i) For each K in $\mathbb{R}^n_+$, $\mathfrak{F}(K)$ is non-empty.

(ii) There is a number $\rho \geqslant 0$ and, for each K in $\mathbb{R}^n_+$, there is a number $B \geqslant 0$ such that, if $(c(\cdot), z(\cdot), k(\cdot))$ is a path from K, then

$$\max\{|c(t)|, |z(t)|\} \leqslant B e^{\rho t} \text{ for } t \geqslant 0, \text{ a.e.} \tag{9}$$

2.2 Examples

In this section we shall provide two examples of the framework described earlier. The examples will play a role in subsequent sections. They have not been chosen for their generality; various linear, as well as nonlinear, multi-sector models that may be accommodated in the framework described earlier may be found in the examples discussed in Magill (1981).

Example 1

This is the well-known one-sector neoclassical growth model of the Cass–Koopmans type (see Cass, 1965; Koopmans, 1965).

There is one good which is both the capital good and the consumption good. Labour is assumed to be constant over time. Let $G: \mathbb{R}_+ \to \mathbb{R}_+$ denote the gross production function; a number δ, satisfying $0 < \delta < \infty$, denotes the constant exponential rate of depreciation of the capital stock; and $w: \mathbb{R}_+ \to \mathbb{R}_+$ denotes a cardinal welfare function. The functions G and w are assumed to satisfy the following properties.

(N1) $G(0) = 0$; G is continuous on $\mathbb{R}_+$; G is twice continuously differentiable on $\mathbb{R}_{++}$; for $k > 0$, $G'(k) > 0$ and $G''(k) \leqslant 0$; there is $k' > 0$ such that, for $k \in (0, k']$, $G'(k) > \delta$; there is $k'' > 0$ such that, for $k \in [k'', \infty)$, $G'(k) < \delta$.

(N2) $w(0) = 0$; w is continuous and concave on $\mathbb{R}_+$; w is twice continuously differentiable on $\mathbb{R}_{++}$; $w'(C) > 0$ for all $C > 0$; $w''(C) < 0$ for all $C > 0$; $w'(C) \to \infty$ as $C \to 0$.

Remark 3. *Defining* $\Omega \equiv \{(c, z, k): k \geqslant 0;\ z \geqslant -\delta k;\ c = w(C)$ *where* $0 \leqslant C \leqslant G(k) - \delta k - z\}$, $\Lambda = \{(z, k): k \geqslant 0;\ G(k) - \delta k \geqslant z \geqslant -\delta k\}$, *and* $u: \Lambda \to \mathbb{R}_+$ *is given by* $u(z, k) = w(G(k) - \delta k - z)$, *for* $(z, k) \in \Lambda$. *It may be verified that Example 1 satisfies (A1)–(A3). Details may be found in Dasgupta and Mitra (1998).*

Example 2

This is a model with one produced good, which serves as both the capital and the consumption good, and an exhaustible resource. Labour is assumed to be constant over time. The model described below is a standard one employed in the literature on optimal allocation of resources over time in the presence of an exhaustible resource (see e.g. Dasgupta and Heal, 1974, 1979; Solow, 1974).

Denote by k_1 the stock of augmentable capital good and by k_2 the stock of the exhaustible resource. A number δ, satisfying $0 \leqslant \delta < \infty$, denotes the constant exponential depreciation rate of augmentable capital. Let $G: \mathbb{R}^2_+ \to \mathbb{R}_+$ denote the gross production function for the capital-cum-consumption good, using the capital input stock k_1 and the flow of exhaustible resource used $(-z_2)$. It is assumed that the flow of resource use cannot exceed a maximum level denoted by $R > 0$. The output $G(k_1, -z_2)$ can be used to replace worn-out capital (if any), δk_1, to augment the capital stock through net investment, z_1, or to provide consumption or utility (c) using a (welfare) function $w: \mathbb{R}_+ \to \mathbb{R}_+$. So the technological possibility set Ω here is defined by

$$\Omega \equiv \{(c, z_1, z_2, k_1, k_2) \equiv (c, z, k) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2_+ : z_1 \geqslant -\delta k_1; -R \leqslant z_2 \leqslant 0;$$
$$\text{there is } 0 \leqslant C \leqslant G(k_1, -z_2) - \delta k_1 - z_1, \text{ such that } c = w(C)\}.$$

The following assumptions are made on G and w.

(R1) (i) $G(0, 0) = G(0, y) = G(x, 0) = 0$ for $x \geqslant 0$ and $y \geqslant 0$.

(ii) G is continuous, concave and nondecreasing on $\mathbb{R}^2_+$, and twice continuously differentiable on $\mathbb{R}^2_{++}$; $G_1(x, y) > 0$, $G_2(x, y) > 0$ and $G_{22}(x, y) < 0$ for $(x, y) \gg 0$.

(R2) $w(0) = 0$, w is continuous and concave on $\mathbb{R}_+$; w is twice continuously differentiable on $\mathbb{R}_{++}$; $w'(C) > 0$ and $w''(C) < 0$ for $C > 0$; $w'(C) \to \infty$ as $C \to 0$.

It may be verified easily that $\Lambda = \{(z_1, z_2, k_1, k_2) : (k_1, k_2) \geqslant 0;\ -R \leqslant z_2 \leqslant 0;\ G(k_1, -z_2) - \delta k_1 \geqslant z_1 \geqslant -\delta k_1\}$ and the formula for $u : \Lambda \to \mathbb{R}_+$ is: $u(z_1, z_2, k_1, k_2) = w(G(k_1, -z_2) - \delta k_1 - z_1)$ for $(z_1, z_2, k_1, k_2) \in \Lambda$. It may also be verified that Example 2 satisfies (A1)–(A3). Details may be found in Dasgupta and Mitra (1998).

2.3 Competitive paths

We shall now elaborate what we mean by a time path of quantities and prices that evolve along an equilibrium of a competitive market economy, from an initial stock K. It would be convenient, for what follows, to introduce the following notation and concepts. Let $p = (p_1, \ldots, p_n)$ denote the prices of the investment goods in terms of the consumption good. Define a function $h : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R}$ by $h(k, p) \equiv \max\{[c + pz]: (c, z) \in S(k)\}$. That is, $h(k, p)$ is the maximum value of output that can be achieved, given input stocks k, at prices p. By Remark 1, $S(k)$ is non-empty and compact, and therefore $h(k, p)$ is well defined. Further, h is convex in p and, since Ω is convex, h is concave in k.

Given our definition of u in (7), we clearly have

$$h(k, p) = \max\{[u(z, k) + pz] : (z, k) \in \Lambda\}. \tag{10}$$

By (A2), for $k \gg 0$, $u(z, k)$ is strictly concave in z, and therefore there is a unique maximizing choice of investment, which solves (10). We can write this maximizing choice of z in (10) as a function $g(k, p)$; that is, $g : \mathbb{R}^n_{++} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$h(k, p) = u(g(k, p), k) + pg(k, p) \text{ and } (g(k, p), k) \in \Lambda. \tag{11}$$

Remark 4. *For (k^0, p^0) such that $k^0 \gg 0$ and $(g(k^0, p^0), k^0)$ in the* interior *of Λ,*

(i) $p^0 + \partial u(g(k^0, p^0), k^0)/\partial z = 0$;

(ii) *by (A2), the function $f(k, p, z) \equiv p + \partial u(z, k)/\partial z$ is defined in an open neighbourhood around $(k^0, p^0, g(k^0, p^0))$, is continuously differentiable, and its derivative matrix with respect to z is non-singular. Therefore, by the implicit function theorem, $g(k, p)$ is continuously differentiable with respect to (k, p) in an open neighbourhood N of (k^0, p^0), and the range of $(g(k, p), k)$, for (k, p) in N, is in an open subset of Λ. It follows that, in this neighbourhood N of (k^0, p^0), h is continuously differentiable and, by the envelope theorem, $\partial h(k, p)/\partial p = g(k, p)$ and $\partial h(k, p)/\partial k = \partial u(g(k, p), k)/\partial k$.*

Let r denote the market rate of interest. A *competitive* path is a path $(c(t), z(t), k(t))$ with associated prices, denoted by absolutely continuous functions of time $(p_1(t), \ldots, p_n(t)) \equiv p(t)$ with $p(t) \geqslant 0$ for $t \geqslant 0$, a.e., satisfying the following two conditions:

$$c(t) + p(t)z(t) = h(k(t), p(t)) \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{12}$$

$$\dot{p}(t) = rp(t) - \partial h(k(t), p(t))/\partial k \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{13}$$

Here, $p(t)$ is the vector of current prices of the investment goods, in terms of the consumption good, prevailing along an equilibrium path, at each date t. Use the notation $(c(t), z(t), k(t), p(t))$ to denote a competitive path with its associated prices. Along a competitive path, for each $t \geqslant 0$, denote $h(k(t), p(t))$ by $Y(t)$; i.e.

$$Y(t) \equiv h(k(t), p(t)) \qquad \text{for } t \geqslant 0. \tag{14}$$

$Y(t)$ is the NNP at time t; it is the maximum value of output achievable from capital stocks $k(t)$ at the prices $p(t)$, and equation (12) says that, along a competitive path, value of output is maximized; i.e.

$$Y(t) = c(t) + p(t)z(t) \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{15}$$

Equation (13) says that asset markets are in equilibrium; that is, no gains can be made by pure arbitrage (see Dorfman *et al.*, 1958; Weitzman, 1976).

2.4 Optimal paths

In this subsection we provide the relevant technical material on optimal paths, which will be useful in our discussion of Weitzman's Rule in the next two sections. Specifically, our discussion covers two important results: the existence of optimal paths, and the price characterization of optimal paths.

For the existence of optimal paths, we need to assume that the rate of interest exceeds the rate of growth. Specifically, we assume the following.

(A4) $\rho < r$ where ρ is the number appearing in the statement of assumption (A3) above.

Under (A3) and (A4), if $(c(\cdot), z(\cdot), k(\cdot))$ is any path from K in $\mathbb{R}^n_+$ then

$$0 \leqslant \int_{t'}^{\infty} e^{-rt}c(t)\,dt \leqslant B\int_{t'}^{\infty} e^{-(r-\rho)t}\,dt = [B/(r-\rho)]e^{-(r-\rho)t'} \qquad \text{for all } t' \in [0, \infty).$$

A path $(c(\cdot), z(\cdot), k(\cdot))$ from K in $\mathbb{R}^n_+$ is called an *optimal* path from K if, for any path $(c'(\cdot), z'(\cdot), k'(\cdot))$ from K, we have $\int_0^\infty e^{-rt}c(t)\,dt \geqslant \int_0^\infty e^{-rt}c'(t)\,dt$.

Under (A1)–(A4), all the conditions for the applicability of Theorem 7.6 of Magill (1981) are met, so that we can state the following result.

Proposition 1. *There exists an optimal path* $(c(\cdot), z(\cdot), k(\cdot))$ *from each* K *in* $\mathbb{R}^n_+$.

For the price characterization of optimal paths, we use the properties of the *value function* associated with the relevant optimization problem and show that the support of the value function provides the present value prices at which an optimal path is *competitive*, and satisfies the capital value transversality condition.

The problem of finding an optimal path within the set of paths from K in $\mathbb{R}^n_+$ is clearly equivalent to the dynamic optimization problem stated below (with the initial date $t' = 0$); and if $(c^*(\cdot), z^*(\cdot), k^*(\cdot))$ is an optimal path from K in $\mathbb{R}^n_+$ then the pair of functions $(z^*(\cdot), k^*(\cdot))$ constitutes a solution of the following problem.

Problem I

$$\left.\begin{aligned} &\max \int_{t'}^{\infty} \mathrm{e}^{-rt} u(z(t),\, k(t))\, \mathrm{d}t \\ &\text{s.t. } (z(t),\, k(t)) \in \Lambda \qquad \text{for } t \in [t',\, \infty), \text{ a.e.} \\ &\quad \dot{k}(t) = z(t) \qquad \text{for } t \in [t',\, \infty), \text{ a.e.} \\ &\quad k(t') = K. \end{aligned}\right\} \tag{16}$$

For each K in $\mathbb{R}^n_+$ and $t' \geqslant 0$, define the *value function* $V(K,\, t')$ by

$$V(K,\, t') \equiv \int_{t'}^{\infty} \mathrm{e}^{-rt} u(z^*(t),\, k^*(t))\, \mathrm{d}t \tag{17}$$

where $(z^*(t),\, k^*(t))$ solves Problem I. Given the stationarity of the utility function and the constraint set Λ, it is clear that

$$V(K,\, t) = \mathrm{e}^{-rt} V(K,\, 0) \qquad \text{for } t \geqslant 0. \tag{18}$$

From the convex structure of the problem, $V(K,\, t)$ is concave in K for each $t \geqslant 0$, so, for each $K \in \mathbb{R}^n_{++}$ there exists a q in $\mathbb{R}^n$ such that $V(K,\, 0) - qK \geqslant V(y,\, 0) - qy$ for all $y \in \mathbb{R}^n_+$.

If $(c(\cdot),\, z(\cdot),\, k(\cdot))$ is a path from K in $\mathbb{R}^n_+$ we shall say that it is *interior* if (i) $(z(t),\, k(t))$ is in the interior of Λ in $\mathbb{R}^n \times \mathbb{R}^n$ for $t \geqslant 0$, a.e., and (ii) $k(t) \gg 0$ for $t \geqslant 0$.

We may now appeal to the Main Theorem of Takekuma (1982, p. 431)[2] to obtain the following result.

Proposition 2. *If* $(c(\cdot),\, z(\cdot),\, k(\cdot))$ *is an optimal path from* K *in* $\mathbb{R}^n_{++}$ *which is interior, then there exists an absolutely continuous function* $q : [0,\, \infty) \to \mathbb{R}^n$ *satisfying*

$$\left.\begin{aligned} &\text{(i)} \quad -q(t) = \mathrm{e}^{-rt} \frac{\partial u}{\partial z}(z(t),\, k(t)) && \textit{for } t \geqslant 0, \text{ a.e.} \\ &\text{(ii)} \quad -\dot{q}(t) = \mathrm{e}^{-rt} \frac{\partial u}{\partial k}(z(t),\, k(t)) && \textit{for } t \geqslant 0, \text{ a.e.} \\ &\text{(iii)} \; V(k(t),\, t) - q(t)k(t) \geqslant V(k,\, t) - q(t)k && \textit{for } k \in R^n_+, \textit{ and } t \geqslant 0. \end{aligned}\right\} \tag{19}$$

We shall refer to $q(t)$ as the present value prices at time t. Note that, since $\partial u(z,\, k)/\partial z \leqslant 0$ for all $(z,\, k)$ in the interior of Λ, we have $q(t) \geqslant 0$ for $t \geqslant 0$, a.e.

Define a price path $(p(t))$ by

$$p(t) \equiv \mathrm{e}^{rt} q(t) \qquad \text{for } t \geqslant 0. \tag{20}$$

Then, we have, for $(z,\, k(t)) \in \Lambda$, $u(z,\, k(t)) + p(t)z - [u(z(t),\, k(t)) + p(t)z(t)] \leqslant [\partial u(z(t),\, k(t))/\partial z]\,(z - z(t)) + p(t)(z - z(t))$ (using concavity of u as a function of z for each $k) = 0$ by using (i) of (19). Thus, we obtain (12).

Next, differentiating (20) with respect to t, and using (ii) of (19), we obtain

2) To appeal to Takekuma's theorem, only part (ii) of the definition of interiority is needed. However, we confine attention to interior paths in the stronger sense of satisfying part (i) as well, in order to simplify arguments.

$\dot{p}(t) - rp(t) = -\partial u(z(t), k(t))/\partial k$ for $t \geqslant 0$, a.e., from which, by Remark 4, (13) follows. We have, therefore, established the following.

Remark 5. *If* $(c(t), z(t), k(t))$ *is an interior path from* K *in* $\mathbb{R}^n_{++}$, *and* $q : [0, \infty) \to \mathbb{R}^n$ *is an absolutely continuous function satisfying parts* (i) *and* (ii) *of* (19), *then* $(c(t), z(t), k(t), p(t))$ *is competitive, where the price path* $p(t)$ *is defined by* (20).

Using Remark 4 and Proposition 2, we note that an interior optimal path $(c(t), z(t), k(t))$ from K in $\mathbb{R}^n_{++}$ is *competitive* at the price path $(p(t))$. Using (iii) of (19), we have $V(k(t), t) - q(t)k(t) \geqslant V(0, t) - q(t)0$. Also $V(k(t), t) \to 0$ as $t \to \infty$ using (18), (A3)(ii) and $r - \rho > 0$. Therefore, we have $\limsup_{t\to\infty} q(t)k(t) \leqslant 0$. Since $q(t)k(t) \geqslant 0$, we can conclude that an interior optimal path also satisfies the *capital value transversality condition*

$$\lim_{t\to\infty} \mathrm{e}^{-rt} p(t)k(t) = 0. \tag{21}$$

Furthermore, it is straightforward to check, using standard methods, that, if a competitive path $(c(t), z(t), k(t), p(t))$ satisfies (21), then $(c(t), z(t), k(t))$ is an optimal path from $k(0)$.

3. Weitzman's Rule

3.1 Welfare significance of national income

The central concept in a study of the welfare significance of national income in a dynamic context is known as *Weitzman's Rule*. Following Weitzman (1976), let us motivate such a study as follows.

Even if we accept the standpoint that it is consumption alone that promotes welfare, and that the ultimate aim of economic activity is to provide consumption goods, the role of net capital formation in a welfare measure is indirect at best. Intuitively, we understand that present net capital formation affects future productive potential, and hence future consumption potential. If welfare is understood to depend on the time path of current and future consumption, then perhaps net investment may be regarded as a proxy for the effect of this investment on welfare through its effect on consumption in the future. It is not quite clear, however, whether an exact relationship between the two exists, even within the confines of a drastically stripped-down scenario where it is possible to focus on this aspect of the problem alone. As Samuelson (1961) argues, a rigorous and meaningful welfare measure is a "wealth like magnitude" such as the present discounted value of future consumption. What connection does this have, if any, with current income concepts? Weitzman (1976) sought to establish that these may be viewed as simply "two sides of the same coin".

Consider the welfare achieved along a competitive path from time t onwards, using t as the origin, that is, the discounted integral of consumptions discounted to time t,

$$\int_t^\infty \mathrm{e}^{-r(s-t)} c(s)\, \mathrm{d}s. \tag{22}$$

We may define a *constant consumption equivalent* of this, namely, a hypothetical constant $\overline{c}(t)$ such that, if consumption achieved at each date s were equal to $\overline{c}(t)$, then the discounted integral of consumptions achieved would be the same as in (22). So $\overline{c}(t)$ is defined by

$$\int_t^{\infty} \mathrm{e}^{-r(s-t)}\overline{c}(t)\,\mathrm{d}s = \int_t^{\infty} \mathrm{e}^{-r(s-t)}c(s)\,\mathrm{d}s. \tag{23}$$

Noting that $\overline{c}(t)$ is a constant on the left-hand side of this equation, we may rewrite this as

$$(\overline{c}(t)/r) = \int_t^{\infty} \mathrm{e}^{-r(s-t)}c(s)\,\mathrm{d}s$$

or as

$$\overline{c}(t) = r\int_t^{\infty} \mathrm{e}^{-r(s-t)}c(s)\,\mathrm{d}s. \tag{24}$$

Weitzman's main proposition seeks to establish that the NNP at time t, $Y(t)$, is this annuity equivalent $\overline{c}(t)$ of the welfare along the path; that is,

$$Y(t) = r\int_t^{\infty} \mathrm{e}^{-r(s-t)}c(s)\,\mathrm{d}s \qquad \text{for } t \geqslant 0. \tag{25}$$

We shall, henceforth, refer to (25) as Weitzman's Rule (WR).

3.2 Weitzman's Rule for optimal paths

For optimal paths, Weitzman's Rule can be derived from the Bellman equation in dynamic programming, when combined with Pontryagin's maximum principle.[3]

Let $(c(t), z(t), k(t))$ be an optimal path from K in $\mathbb{R}^n_{++}$. If the optimal path is interior and $z(t)$ is a piecewise-continuous function of t, then (by Corollary 1, p. 731, of Benveniste and Scheinkman, 1979) V is continuously differentiable at $k(t)$ for each $t \geqslant 0$, and $V'(k(t), 0) = -\partial u(z(t), k(t))/\partial z$ for $t \geqslant 0$. Then Bellman's equation of optimality (see e.g. Intriligator, 1971; Sorger, 1992) yields for $t \geqslant 0$

$$-V'(k(t), 0)\dot{k}(t) = c(t) - rV(k(t), 0). \tag{26}$$

Using (18), (19) and (20), we know that the derivative of the value function, $V'(k(t), 0)$, is equal to the price, $p(t)$, of the investment good (where $(p(t))$ is the price path at which the optimal path $(c(t), z(t), k(t))$ is competitive). This yields, for $t \geqslant 0$, $-p(t)\dot{k}(t) = c(t) - rV(k(t), 0)$, from which, upon rewriting, we obtain $c(t) + p(t)\dot{k}(t) = r\int_t^{\infty} \mathrm{e}^{-r(s-t)}c(s)\,\mathrm{d}s$, which is Weitzman's Rule.

3.3 Weitzman's Rule for competitive paths and a transversality condition

While it is a point of interest that Weitzman's Rule is valid for optimal paths, it would be of far more interest if indeed it were valid for any competitive path. In fact, Weitzman's discussion of the problem, providing a motivation, seems to be along precisely such lines. Given an infinite horizon path satisfying competitive equilibrium

3) This approach has been followed by several authors, including Hartwick (1994) and Kemp and Long (1982). For alternate derivations of this result, see Asheim (1994).

conditions in a perfect-foresight framework, the current prices and quantities carry information about the future prices and quantities. Since current net investment affects future consumption possibilities, intuitively there are grounds to explore if there is a clear-cut relation between current NNP and the flow of consumption along the path.

Our investigation of this issue leads to a characterization of competitive paths that satisfy Weitzman's Rule. We find that these paths can be characterized by a transversality condition involving the present value of net *investment*.[4] Specifically, we show that a competitive path $(c(t), z(t), k(t), p(t))$ satisfies WR if and only if

$$e^{-rt}p(t)\dot{k}(t) \to 0 \text{ as } t \to \infty.$$

Note that the condition[5] is apparently different from the usual (capital value) transversality condition characterizing optimal paths, which states that the present value of *capital*, $e^{-rt}p(t)k(t)$, goes to zero as $t \to \infty$.

We first state a technical lemma which will be needed in the proof of Theorem 1 below.

Lemma 1. *If $(c(t), z(t), k(t), p(t))$ is a competitive path from K in $\mathbb{R}^n_{++}$, which is interior, then the function $Y(t)$, defined by (14), is an absolutely continuous function of t.*

Proof. Let $0 \leqslant a < b < \infty$ be given. For $t \in [a, b]$, we have $Y(t) = h(k(t), p(t))$. Now $(k(t), p(t))$ are continuous on $[a, b]$, so we can find $0 \leqslant m \leqslant M < \infty$, such that, for all $t \in [a, b]$, $m \leqslant k_i(t) \leqslant M$ for $i = 1, \ldots, n$ and $0 \leqslant p_i(t) \leqslant M$ for $i = 1, \ldots, n$. Since the competitive path is interior, we may choose $m > 0$. Thus, $E \equiv [me, Me]$ is a compact subset in the interior of $\mathbb{R}^n_+$, where $e = (1, 1, \ldots, 1)$ in $\mathbb{R}^n$.

Let t_1, t_2 be arbitrary points in $[a, b]$. Then, $Y(t_2) - Y(t_1) = h(k(t_2), p(t_2)) - h(k(t_1), p(t_1)) = h(k(t_2), p(t_2)) - h(k(t_1), p(t_2)) + h(k(t_1), p(t_2)) - h(k(t_1), p(t_1))$. Thus, we have:

$$\text{(i) } |Y(t_2) - Y(t_1)| \leqslant |h(k(t_2), p(t_2)) - h(k(t_1), p(t_2))| + |h(k(t_1), p(t_2)) - h(k(t_1), p(t_1))|.$$

The function $H(k) = h(k, p(t_2))$ is a concave function on $\mathbb{R}^n_+$, and therefore is Lipschitz on the compact subset E in the interior of $\mathbb{R}^n_+$, with Lipschitz constant, $L_1 > 0$. Thus, we have:

$$\text{(ii) } |h(k(t_2), p(t_2)) - h(k(t_1), p(t_2))| \leqslant L_1|k(t_2) - k(t_1)|.$$

The function $G(p) = h(k(t_1), p)$ is a convex function on $\mathbb{R}^n$, and therefore is Lipschitz on the compact subset $[0, Me]$ in the interior of $\mathbb{R}^n$, with Lipschitz constant, $L_2 > 0$. Thus, we have:

$$\text{(iii) } |h(k(t_1), p(t_2)) - h(k(t_1), p(t_1))| \leqslant L_2|p(t_2) - p(t_1)|.$$

4) This result has been derived in alternative frameworks by Skiba (1978) and Dechert and Nishimura (1983), among others. However, they do not interpret this result as central to understanding the welfare significance of national income (following the lines of Weitzman) as we do.

5) Since $k(t)$ is differentiable almost everywhere, the limit condition ought to be interpreted in an "almost everywhere" sense; more precisely, if $x : [0, \infty) \to \mathbb{R}$, then by the statement "$x(t) \to x$ as $t \to \infty$" we mean that, "given any $\varepsilon > 0$, there is $T \geqslant 0$ such that for $t \geqslant T$, a.e., $|x(t) - x| < \varepsilon$".

Given any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $a_1, b_1, \ldots, a_r, b_r$ are numbers satisfying $a \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_r < b_r \leqslant b$ and $\sum_{j=1}^{r}(b_j - a_j) < \delta$, then, for all $i = 1, \ldots, n$, $\sum_{j=1}^{r}|k_i(b_j) - k_i(a_j)| < (\varepsilon/2nL_1)$ and for all $i = 1, \ldots, n$, $\sum_{j=1}^{r}|p_i(b_j) - p_i(a_j)| < (\varepsilon/2nL_2)$, since k and p are absolutely continuous on $[a, b]$. Thus, using (i), (ii) and (iii) above, we have

$$\sum_{j=1}^{r}|Y(b_j) - Y(a_j)| \leqslant L_1(\varepsilon/2L_1) + L_2(\varepsilon/2L_2) = \varepsilon,$$

which proves that $Y(t)$ is absolutely continuous on $[a, b]$. ∎

Theorem 1. *If* $(c(t), z(t), k(t), p(t))$ *is a competitive path from* K *in* $\mathbb{R}^n_{++}$, *which is interior, then*

(i) $\dot{Y}(t) = r[Y(t) - c(t)]$ *for* $t \geqslant 0$, *a.e.*;
(ii) *for every* $t \geqslant 0$, $\lim_{T\to\infty} e^{-r(T-t)}p(T)\dot{k}(T)$ *exists and*

$$r\int_t^\infty e^{-r(s-t)}c(s)\,ds = Y(t) - \lim_{T\to\infty} e^{-r(T-t)}p(T)\dot{k}(T).$$

Proof. Recall that $g(k, p)$ denotes the solution to the maximization problem in (10) for $k \gg 0$. Using condition (12) for a competitive path, we have, for $t \geqslant 0$, a.e., $g(k(t), p(t)) = z(t)$, and, since the path is interior, $(g(k(t), p(t)), k(t))$ is in the interior of Λ, so, by Lemma 1, $Y(t)$ is absolutely continuous. Therefore, for $t \geqslant 0$, a.e., $Y(t)$ is differentiable; also $k(t)$, $p(t)$ are differentiable, and, by Remark 4, $h(k, p)$ is continuously differentiable at $(k(t), p(t))$, so that

$$\begin{aligned}\dot{Y}(t) &= [\partial h(k(t), p(t))/\partial k]\dot{k}(t) + [\partial h(k(t), p(t))/\partial p]\dot{p}(t)\\ &= [\partial h(k(t), p(t))/\partial k]\dot{k}(t) + g(k(t), p(t))\dot{p}(t) \qquad \text{for } t \geqslant 0, \text{ a.e.}\end{aligned}$$

Thus, we obtain

$$\dot{Y}(t) = [\partial h(k(t), p(t))/\partial k]\dot{k}(t) + \dot{k}(t)\dot{p}(t) \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{27}$$

Combining condition (13) for a competitive path with (15) and (27), $\dot{Y}(t) = rp(t)\dot{k}(t) = r[Y(t) - c(t)]$ for $t \geqslant 0$, a.e., which establishes (i).

Using (i), we have for $s \in [t, \infty)$, a.e., $re^{-r(s-t)}c(s) = e^{-r(s-t)}[rY(s) - \dot{Y}(s)]$. Now, differentiating $e^{-r(s-t)}Y(s)$, we get, for $s \in [t, \infty)$, a.e.,

$$\frac{d}{ds}[e^{-r(s-t)}Y(s)] = (-r)e^{-r(s-t)}Y(s) + e^{-r(s-t)}\dot{Y}(s),$$

so that

$$re^{-r(s-t)}c(s) = -\frac{d}{ds}[e^{-r(s-t)}Y(s)] \text{ for } s \in [t, \infty), \text{ a.e.} \tag{28}$$

By Lemma 1, $Y(t)$ is an absolutely continuous function of t and therefore is an indefinite integral. So, integrating (28) from $s = t$ to $s = T > t$, we get

$$r\int_t^T e^{-r(s)-t)}c(s)\mathrm{d}s = -[e^{-r(s-t)}Y(s)]_t^T = Y(t) - e^{-r(T-t)}Y(T) \qquad \text{for } t \geqslant 0,\ T \geqslant t. \tag{29}$$

Using (A3), the left-hand side of (29) has a limit as $T \to \infty$, and so

$$\lim_{T\to\infty} e^{-r(T-t)}Y(T) \text{ exists.} \tag{30}$$

Using (A3) again, we have $\lim_{T\to\infty} e^{-r(T-t)}c(T) = 0$. And since, by (15), $e^{-r(T-t)}Y(T) = e^{-r(T-t)}c(T) + e^{-r(T-t)}p(T)\dot{k}(T)$, for $T \geqslant 0$, a.e., we obtain

$$\lim_{T\to\infty} e^{-r(T-t)}p(T)\dot{k}(T) \text{ exists and} \qquad \lim_{T\to\infty} e^{-r(T-t)}p(T)\dot{k}(T) = \lim_{T\to\infty} e^{-r(T-t)}Y(T). \tag{31}$$

Using (31) in (29) establishes (ii). ■

A straightforward consequence of Theorem 1 is the following characterization result.

Corollary 1. *A competitive path* $(c(t), z(t), k(t), p(t))$ *from* K *in* $\mathbb{R}^n_{++}$, *which is interior, satisfies*

$$r\int_t^\infty e^{-r(s-t)}c(s)\,\mathrm{d}s = Y(t) \qquad \text{for } t \geqslant 0 \tag{32}$$

if and only if the investment value transversality condition is satisfied, i.e. if

$$\lim_{t\to\infty} e^{-rt}p(t)\dot{k}(t) = 0. \tag{33}$$

3.4 An example of a competitive path which violates Weitzman's Rule

Our characterization prompts us to examine whether Weitzman's Rule holds for every competitive path; that is, whether every competitive path automatically satisfies the investment value transversality condition.

Example 3

We construct a concrete example of the one-sector neoclassical growth model described earlier (see Example 1 in Section 2.2), where a competitive path violates the investment value transversality condition and consequently also Weitzman's Rule.

Define the net output function, F, by $F(k) \equiv G(k) - \delta k$ for $k \geqslant 0$, where the depreciation rate $\delta = 0.5$, and the gross output function, G, is defined by

$$G(k) \equiv \begin{cases} 0.75 + 0.25k - 0.75(1-k)^3 & \text{for } 0 \leqslant k \leqslant 1 \\ 0.75 + 0.25k & \text{for } k \geqslant 1 \end{cases}.$$

Define the welfare function $w\colon R_+ \to R_+$ by: $w(C) \equiv 2C^{0.5}$ for $C \geqslant 0$. It is straightforward to verify that (N1) and (N2) of Example 1 are satisfied.

We now define a path in this model that will turn out to be competitive, but will *not* satisfy Weitzman's Rule.

Define the functions $m(t) \equiv 1.2e^{-0.25t} - 0.2e^{-t}$ for $t \geqslant 0$; $k(t) \equiv 3 - m(t)$ for $t \geqslant 0$; $C(t) \equiv F(k(t)) - \dot{k}(t)$ for $t \geqslant 0$. It may be verified that $\dot{k}(t) > 0$ and $C(t) > 0$ for $t \geqslant 0$, and, defining $z(t) \equiv \dot{k}(t)$ and $c(t) \equiv w(C(t))$ for $t \geqslant 0$, we note that $(c(t), z(t), k(t))$ is an interior path from $k(0) = 2$.

Now, define the interest rate r as $r \equiv 0.25$, and the price functions $p(t)$, $q(t)$ by $p(t) \equiv w'(C(t))$, $q(t) \equiv e^{-rt}w'(C(t))$ for $t \geqslant 0$. It may then be verified that parts (i) and (ii) of (19) are satisfied, and so, by Remark 5, $(c(t), z(t), k(t), p(t))$ is competitive. Also, it may be shown that for this competitive path $q(t)\dot{k}(t)$ converges to a positive limit as $t \to \infty$, so that the investment value transversality condition does not hold. Using Corollary 1, this implies that Weitzman's Rule is not satisfied by this competitive path. (For details, readers are referred to Dasgupta and Mitra, 1998.)

4. Exhaustible resources and sustainable development

4.1 Exhaustible resources and Weitzman's Rule

An interesting observation, in the context of a standard optimal growth model with an exhaustible resource as an essential factor of production, is that *all* competitive paths satisfy the investment value transversality condition and, therefore, Weitzman's Rule.

We have described this framework earlier in Example 2 of Section 2.2. Readers should refer to that description for relevant details. Denoting by x and y the first and second arguments of the production function G, where x stands for the input of the augmentable capital good and y stands for the input of exhaustible resource use, we also assume, in addition to (R1) and (R2) of Section 2.2, that the share of the exhaustible resource in the production of the first good is bounded away from zero. More precisely, we assume that

(R3) $$\alpha \equiv \inf_{(x,y)\gg 0} [y(\partial G(x, y)/\partial y)/G(x, y)] > 0.$$

Finally, we also assume that the marginal productivity of capital diverges to infinity as the capital–resource ratio goes to zero. More precisely, we assume the following.

(R4) Given any number μ, there is $\varepsilon > 0$ such that $(x, y) \gg 0$,

$$0 < y \leqslant R \text{ and } (x/y) \leqslant \varepsilon \text{ implies that } G_1(x, y) \geqslant \mu.$$

It is straightforward to verify that the function $G(x, y) = x^{\beta}y^{\alpha}$ where $\alpha > 0$, $\beta > 0$ and $\alpha + \beta \leqslant 1$ is an example of a production function which satisfies the assumptions (R1), (R3) and (R4).

Denote by v the inverse of w. In terms of the notation of Section 2, the technological possibility set is then given by

$$\Omega = \{(c, z, k) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2_+ : z_1 \geqslant -\delta k_1, \qquad -R \leqslant z_2 \leqslant 0,$$
$$G(k_1, -z_2) \geqslant \delta k_1 + z_1 + v(c)\}.$$

Before we state and prove the main result of this section, we need a lemma, which provides a uniform positive lower bound on the marginal product of the resource, when the capital–resource ratio has a uniform positive lower bound.

Lemma 2. *Given any three numbers* $\bar{x}, \bar{y}$ *and* $\varepsilon > 0$, *there is* $\mu > 0$ *such that* $G_2(x, y) \geqslant \mu$ *for any* (x, y) *satisfying* $0 \leqslant x \leqslant \bar{x}$, $0 < y \leqslant \bar{y}$, *and* $(x/y) \geqslant \varepsilon$.

Proof. Define two sets A and B as follows:

$$A \equiv \{(x, y) \in \mathbb{R}^2_+ : \ 0 \leqslant x \leqslant \bar{x}, 1 \leqslant y \leqslant \bar{y}, (x/y) \geqslant \varepsilon\}$$

$$B \equiv \{(x, y) \in \mathbb{R}^2_+ : \ 0 \leqslant x \leqslant \bar{x}, 0 < y \leqslant 1, (x/y) \geqslant \varepsilon\}.$$

For any (x, y) in B, $(x, y) \gg 0$. So $G_2(x, y) \geqslant \alpha G(x, y)/y \geqslant \alpha G((x/y), 1)$ (using the fact that, as a consequence of concavity of G and $G(0, 0) \geqslant 0$, $G(\lambda x, \lambda y) \leqslant \lambda G(x, y)$ for (x, y) in $\mathbb{R}^2_+$ and $\lambda \geqslant 1) \geqslant \alpha G(\varepsilon, 1)$, since G is increasing and $(x/y) \geqslant \varepsilon > 0$. Now, if A is empty, we may choose $\mu \equiv \alpha G(\varepsilon, 1) > 0$ and we are done. If A is non-empty, note that $(x, y) \in A$ implies $x \geqslant \varepsilon y \geqslant \varepsilon$. So A is a compact set in $\mathbb{R}^2_{++}$. Using the continuity of $G_2(x, y)$ on $\mathbb{R}^2_{++}$, it attains a minimum value, call it $\bar{\varepsilon}$; then, since $G_2(x, y) > 0$ for all (x, y) in A, $\bar{\varepsilon} > 0$. We may now define $\mu \equiv \text{Min}\{\alpha G(\varepsilon, 1), \bar{\varepsilon}\} > 0$. Observing that (x, y) satisfying the conditions of the lemma must be in either A or B, the proof is complete. ■

Theorem 2. *If* $(c(\cdot), z(\cdot), k(\cdot), p(\cdot))$ *is a competitive path from* $(K_1, K_2) \gg 0$ *which is interior, then it satisfies Weitzman's Rule.*

Proof. Suppose that $(c(\cdot), z(\cdot), k(\cdot), p(\cdot))$ is a competitive path from $(K_1, K_2) \gg 0$ which is an interior path. Then,

$$h(k(t), p(t)) = \max w[G(k_1(t), -z_2) - \delta k_1(t) - z_1] + p_1(t)z_1 + p_2(t)z_2$$

$$\text{s.t. } (z_1, z_2, k_1(t), k_2(t)) \in \Lambda.$$

Using first-order conditions for an interior maximum, we get

$$w'[v(c(t))](-1) + p_1(t) = 0 \qquad \text{for } t \geqslant 0, \ \text{a.e.}; \tag{34}$$

$$w'[v(c(t))][G_2(k_1(t), -z_2(t))(-1)] + p_2(t) = 0 \qquad \text{for } t \geqslant 0, \ \text{a.e.} \tag{35}$$

Also, by the envelope theorem,

$$\frac{\partial h(k(t), p(t))}{\partial k_1} = w'[v(c(t))][G_1(k_1(t), -z_2(t)) - \delta] \qquad \text{for } t \geqslant 0, \ \text{a.e.}, \tag{36}$$

$$\frac{\partial h(k(t), p(t))}{\partial k_2} = 0 \qquad \text{for } t \geqslant 0, \ \text{a.e.} \tag{37}$$

We can use (37) to study the behaviour of the price of the exhaustible resource over time. Using (37) in the competitive condition (13), we get

$$\dot{p}_2(t) = rp_2(t) \qquad \text{for } t \geqslant 0, \ \text{a.e.}, \tag{38}$$

so that the current-value price of the exhaustible resource must rise exponentially at the interest rate, r:

$$p_2(t) = p_2(0)\mathrm{e}^{rt} \qquad \text{for } t \geqslant 0. \tag{39}$$

Defining $q(t) \equiv p(t)\mathrm{e}^{-rt}$ for $t \geqslant 0$, (39) implies that the present-value price of the exhaustible resource is constant over time:

$$q_2(t) = p_2(0) \qquad \text{for } t \geqslant 0. \tag{40}$$

Similarly, we can use (34)–(36) to study the behaviour of the price of the augmentable capital good. Specifically, (34) and (35) can be used to relate the present-value price of the augmentable capital good to the marginal product of the exhaustible resource as follows:

$$q_1(t) = p_2(0)/G_2(k_1(t), -z_2(t)) \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{41}$$

Equation (36) can also be used to relate the rate of change of the present-value price of the augmentable capital good to the marginal product of capital. To see this, note that, since $w'(\cdot) > 0$, $G_1(\cdot) > 0$, and $G_2(\cdot) > 0$, we have $q_1(t) > 0$ and $q_2(t) > 0$ for $t \geqslant 0$, a.e. Also, $q_1(t) = \mathrm{e}^{-rt} p_1(t)$ for $t \geqslant 0$, so, by differentiating this equation with respect to t, $\dot{q}_1(t) = -r\mathrm{e}^{-rt} p_1(t) + \mathrm{e}^{-rt}\dot{p}_1(t) = -rq_1(t) + \mathrm{e}^{-rt}[rp_1(t) - [\partial h(k(t), \ p(t))/\partial k_1]] = -rq_1(t) + rq_1(t) - \mathrm{e}^{-rt}[\partial h(k(t), \ p(t))/\partial k_1]$. Thus, using (36),

$$\dot{q}_1(t) = -\mathrm{e}^{-rt} w'(v(c(t)))[G_1(k_1(t), -z_2(t)) - \delta] \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{42}$$

We will now put an upper bound on the present value of investment in terms of the exhaustible resource use. Use (41) and assumption (R3) to obtain for $t \geqslant 0$, a.e.:

$$q_1(t)v(c(t)) + q_1(t)z_1(t) \leqq q_1(t)G(k_1(t), -z_2(t))$$

$$= \frac{G(k_1(t), -z_2(t))(-z_2(t))}{G_2(k_1(t), -z_2(t))(-z_2(t))} p_2(0) \leqslant [-z_2(t)]p_2(0)/\alpha.$$

Since $v(c(t)) \geqslant 0$, we get

$$q_1(t)z_1(t) \leqslant [-z_2(t)]p_2(0)/\alpha \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{43}$$

Adding $q_2(t)z_2(t) = p_2(0)z_2(t)$ to both sides of the inequality in (43), we get

$$q_1(t)z_1(t) + q_2(t)z_2(t) \leqslant [-z_2(t)]p_2(0)[(1-\alpha)/\alpha] \qquad \text{for } t \geqslant 0, \text{ a.e.} \tag{44}$$

For notational ease, in the rest of the proof we shall use $z(t)$ and $\dot{k}(t)$ interchangeably, it being understood that statements involving $z(t)$ are meant to hold for the set of t, where $\dot{k}(t) = z(t)$, which is a set whose complement is a set of measure zero.

By Theorem 1 (ii), we know that $\lim_{t\to\infty} q(t)z(t)$ exists, and we now claim that, in the limit, the present value of investment is non-positive; i.e.

$$\lim_{t\to\infty} q(t)z(t) \leqslant 0. \tag{45}$$

Suppose, contrary to (45), there is $\theta > 0$ such that $\lim_{t\to\infty} q(t)z(t) = \theta$. Then, we can find T large enough so that, for $t \geqslant T$, a.e., we have $q(t)z(t) \geqslant (\theta/2)$. This means, by using (44), that

$$[-z_2(t)]p_2(0)[(1-\alpha)/\alpha] \geqslant (\theta/2) \qquad \text{for } t \geqslant T, \text{ a.e.} \tag{46}$$

Clearly, we can pick $N > T$ so that

$$(N - T)(\theta/2) > k_2(0)p_2(0)[(1-\alpha)/\alpha]. \tag{47}$$

However, using (46), we get

$$(\theta/2)(N - T) \leqslant \int_T^N [-z_2(t)]p_2(0)[(1-\alpha)/\alpha]\,\mathrm{d}t = p_2(0)[(1-\alpha)/\alpha][k_2(T) - k_2(N)],$$

so that $(\theta/2)(N - T) \leqslant p_2(0)[(1 - \alpha)/\alpha]k_2(0)$, which contradicts (47) and establishes (45).

It remains to show that the limiting present value of investment is actually zero:

$$\lim_{t\to\infty} q(t)z(t) = 0. \tag{48}$$

Suppose that (48) is violated. Then, in view of (45), there is $\theta > 0$, such that $\lim_{t\to\infty} q(t)z(t) = -\theta$. So, we can find T sufficiently large that

$$q_1(t)z_1(t) + q_2(t)z_2(t) < -(\theta/2) \qquad \text{for } t \geqslant T, \text{ a.e.} \tag{49}$$

We now break up our analysis into two cases, depending on whether capital is non-depreciating ($\delta = 0$) or partly depreciating ($0 < \delta$).

Case 1 ($\delta = 0$)

In this case, $z_1(t) \geqslant -\delta k_1(t) = 0$, and $q_2(t) = q_2(0)$ for all t, so that $z_2(t) \leqslant -(\theta/2q_2(0))$ for $t \geqslant T$, a.e. Thus, $\int_T^S(-z_2(t))\,\mathrm{d}t \geqslant (\theta/2q_2(0))(S - T)$ for $S \in [T, \infty)$. This contradicts $\int_0^\infty(-z_2(t))\,\mathrm{d}t \leqslant k_2(0)$ and establishes (48).

Case 2 ($0 < \delta$)

In this case, we shall show that the augmentable capital stock and its present value price are uniformly bounded over time. We shall first show that the time path $(k_1(\cdot))$ is bounded. Since $G(1, 0) = 0$, we can find $\gamma > 0$ such that $G(1, \gamma) \leqslant (\delta/2)$. Let $E \equiv \text{Max}\{1, (R/\gamma)\}$. For $t \geqslant 0$, a.e., if $k_1(t) \geqslant E$, then $[-z_2(t)/k_1(t)] \leqslant [R/k_1(t)] \leqslant \gamma$, and so, since $k_1(t) \geqslant E \geqslant 1$, we have (using the fact that, as a consequence of concavity of G and $G(0, 0) \geqslant 0$, $G(\lambda x, \lambda y) \geqslant \lambda G(x, y)$ for (x, y) in $\mathbb{R}_+^2$ and $0 \leqslant \lambda \leqslant 1$) $[1/k_1(t)]G(k_1(t), -z_2(t)) \leqslant G(1, [-z_2(t)/k_1(t)]) \leqslant G(1, \gamma) \leqslant (\delta/2)$. Thus, using $z_1(t) \leqslant G(k_1(t), -z_2(t)) - \delta k_1(t)$, we obtain

$$z_1(t) \leqslant -(\delta/2)k_1(t) \qquad \text{whenever } k_1(t) \geqslant E, \text{ for } t \geqslant 0, \text{ a.e.} \tag{50}$$

We now claim that

$$k_1(t) < \overline{K} \qquad \text{for } t \geqslant 0, \tag{51}$$

where $\overline{K} = \text{Max}\{2K_1, 2E\}$. Clearly, $k_1(0) = K_1 < \overline{K}$. To establish (51), suppose on the contrary that $k_1(t) \geqslant \overline{K}$ for some t. Then, by continuity of $k_1(\cdot)$, we can find t_0, t_1 such that $0 < t_0 < t_1$ and $E < k_1(t_1) < \overline{K} \leqslant k_1(t_1)$ for $t \in [t_0, t_1)$. But then, by (50), $\dot{k}_1(t) < 0$ for $t \in [t_0, t_1)$, a.e., and so $k_1(t) < k_1(t_0)$, a contradiction.

We will next show that the price path of present values $(q_1(\cdot))$ is bounded. By (R4), we can find $\varepsilon > 0$ such that, for $(x, y) \gg 0$ satisfying $y \leqslant R$ and $(x/y) \leqslant \varepsilon$, we have $G_1(x, y) \geqslant 2\delta$. Next, by Lemma 2, given $\overline{K}$, R and ε, there is $\mu > 0$ such that $G_2(x, y) \geqslant \mu$ whenever $(x, y) \gg 0$ satisfy $x \leqslant \overline{K}$, $y \leqslant R$ and $(x/y) \geqslant \varepsilon$. For $t \geqslant 0$, a.e., since $0 < k_1(t) \leqslant \overline{K}$ and $0 < [-z_2(t)] \leqslant R$, we have either (i) $[k_1(t)/(-z_2(t))] \geqslant \varepsilon$ and, therefore, $G_2(k_1(t), -z_2(t)) \geqslant \mu$, or (ii) $[k_1(t)/(-z_2(t))] < \varepsilon$, and therefore $G_1(k_1(t), -z_2(t)) \geqslant 2\delta$. Using (41), we have $q_1(t) \leqslant [p_2(0)/\mu]$ in case (i); and, using (42), we have $\dot{q}_1(t) < 0$ in case (ii). To summarize, we have for $t \geqslant 0$, a.e.,

$$\text{either} \qquad \text{(i) } q_1(t) \leqslant [p_2(0)/\mu] \qquad \text{or} \qquad \text{(ii) } \dot{q}_1(t) < 0. \tag{52}$$

We now claim that

$$q_1(t) \leqslant \overline{q}_1 \qquad \text{for } t \geqslant 0, \tag{53}$$

where $\overline{q}_1 = \text{Max}\{q_1(0),\ p_2(0)/\mu\}$. Clearly, $q_1(0) \leqslant \overline{q}_1$. To establish (53), suppose to the contrary that $q_1(t) > \overline{q}_1$ for some $t = t_1$, say. Then, by continuity of $q_1(\cdot)$, we can find t_0 such that $0 \leqslant t_0 < t_1$ and $q_1(t) > \overline{q}_1 = q_1(t_0)$ for all $t \in (t_0, t_1]$, a.e. But then by definition of $\overline{q}_1$ and (52), $\dot{q}_1(t) < 0$ for $t \in (t_0, t_1]$, a.e., and so $q_1(t_1) < q_1(t_0)$, a contradiction.

Using the bounds on the augmentable capital stock and its present value price, we now show that (49) leads to a contradiction, thereby establishing (48).

Since $\int_0^\infty [-z_2(t)]\,\mathrm{d}t \leqslant K_2$ and $z_2(t) \leqslant 0$ for $t \geqslant 0$, a.e., the set $M_1 \equiv \{t \in [0, \infty)$: $[-z_2(t)] \geqslant [\theta/4q_2(0)]\}$ has Lebesgue measure $\upsilon(M_1) \leqslant [4K_2q_2(0)/\theta]$. Denote the complement of M_1 in $[0, \infty)$ by M_2. Then, for $t \in M_2$, $[-z_2(t)] < [\theta/4q_2(0)]$. Since $q_2(t) = q_2(0)$ for all t, this means that $-q_2(t)z_2(t) \leqslant (\theta/4)$. Using (49), we have $q_1(t)z_1(t) < -(\theta/4)$ for $t \in [T, \infty) \cap M_2$, a.e. We can now use (53) to obtain $z_1(t) < -(\theta/4\overline{q}_1)$ for $t \in [T, \infty) \cap M_2$, a.e.

Using (51), for $t \geqslant 0$, a.e., $G(k_1(t), -z_2(t)) \leqslant G(\overline{K}, R)$, so that $z_1(t) \leqslant G(\overline{K}, R)$. So, for any $S > T$, we have

$$\begin{aligned} k_1(S) = k_1(T) + \int_T^S z_1(t)\,\mathrm{d}t = k_1(T) + \int_{M_1 \cap [T,S]} z_1(t)\,\mathrm{d}t \\ + \int_{M_2 \cap [T,S]} z_1(t)\,\mathrm{d}t \leqslant k_1(T) + \upsilon(M_1 \cap [T, S])G(\overline{K}, R) \\ - (\theta/4\overline{q}_1)\upsilon(M_2 \cap [T, S]). \end{aligned}$$

Since $\upsilon(M_1)$ is finite, $\upsilon(M_2 \cap [T, S]) \to \infty$ as $S \to \infty$. Thus, $k_1(S) < 0$ for S sufficiently large, a contradiction.

We may now apply Corollary 1 to conclude that the given competitive path satisfies Weitzman's Rule. ■

4.2 National income accounts and sustainable development

In the previous subsection we have shown that, for competitive paths (in a model with exhaustible resources), NNP is a measure of welfare. However, note that the investment component is all-inclusive; the concept of the capital stock, in Weitzman's formulation, is very broad. Any factor that influences the production possibilities, the menu of possible consumption and investment, which changes over time, and whose level is a matter of economic choice, is included in this. The net change in such a stock, valued at the appropriate shadow price of that stock, ought to be included in the investment component in order that NNP properly measures the welfare achieved along the path, namely the annuity equivalent of future discounted sum of utility flow. If mineral resources, whose stocks can only be used up but not augmented, are used in the production processes, then the value of the change in these stocks ought to be included.

Under current national income accounting practices, in terms of the model of the previous subsection, the NNP at date t would be measured by

$$c(t) + p_1(t)z_1(t) = \hat{Y}(t).$$

This is to be contrasted with NNP in Weitzman's sense, which would be

$$c(t) + p_1(t)z_1(t) + p_2(t)z_2(t) = Y(t).$$

Exhaustible resource stocks, like any other "produced" capital stocks, are part of the productive stocks, and the value of net investment in these (in this case a negative one, since $z_2(t) \leqslant 0$) is also accounted for in this measure of NNP. This measure is typically smaller than the currently used measure, and it is only this new measure of NNP that accurately reflects the future welfare (discounted sum of future consumptions) of this dynamic economy. (For a more detailed discussion, see Dasgupta, 1990; Hartwick, 1990 and Maler, 1991.)

Weitzman's Rule makes it very tempting to link NNP with some suitable notion of sustainable welfare.[6] The intuition behind seeking a connection between NNP and welfare, in the sense of the discounted sum of utilities, is that current net investment adds to future consumption potential. This could be rephrased, adopting a slightly different viewpoint. Gross domestic product is not suitable as an indicator of national wellbeing, because, if two nations produce the same total final output for consumption, but one of them does so after making provisions to replace or repair worn-out equipment used in production while the other does not, then we would say that the former nation is providing for its citizens better than the latter. The former can sustain this level of wellbeing, while the latter would be unable to do so, since it is running down its productive capacity. So a depreciation charge needs to be deducted from GNP to arrive at a suitable measure of wellbeing. Loosely speaking, the standpoint in this line of reasoning is that a proper notion of income, which is at one's disposal for consumption, is like an interest income on a fixed stock of capital. This is what can be consumed without depleting future earning and spending power.

If we now also adopt the standpoint that what is meant by sustainable development is not the preservation of specific things—natural resources, environmental riches, etc.—but rather a policy under which future generations have the possibility of ensuring for themselves and their successors at least the same level of wellbeing as the current generation, then it seems natural to suggest that a policy of sustainable development should be one that adequately provides for depreciation of capital in a broad sense. If non-renewable mineral resource stocks are depleted to enable current productive activities to be carried out, then this "depreciation" of stocks should be made up in the only possible way, i.e. through "appropriate" investment in produced capital goods which can substitute for the resource in production.[7]

Specifically, if (following Weitzman, 1995), by a path of sustainable development, we mean that welfare, as measured by the discounted sum of consumption criterion, of future generations is at least as much as the welfare of the current generation, then the value of net investment is an indicator of whether or not the path considered is one of sustainable development. The welfare of future generations can be represented by the constant consumption equivalent of the future flow of consumption, and this is

6) Sustainable welfare, in the sense of maintaining a minimum positive level of consumption, when an exhaustible resource is an essential factor of production, has been examined by Solow (1974), Hartwick (1977) and Cass and Mitra (1991).

7) The concept of sustainable development, in a model with exhaustible resource constraints, has been interpreted and studied in various ways; see e.g. Solow (1986, 1992), Hartwick (1994), and Asheim (1994).

measured by NNP. This is larger than current consumption if and only if the value of net investment is positive.

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